Int J Theor Phys (2008) 47: 2655–2662
DOI 10.1007/s10773-008-9702-5

A Quantum Mechanical Relation Connecting Time, Temperature, and Cosmological Constant of the Universe: Gamow'S Relation Revisited as a Special Case

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Received: 9 November 2007 / Accepted: 10 March 2008 / Published online: 18 March 2008

Abstract Considering our expanding universe as made up of gravitationally interacting particles which describe particles of luminous matter, dark matter and dark energy which is represented by a repulsive harmonic potential among the points in the flat 3-space and incorporating Mach's principle into our theory, we derive a quantum mechanical relation connecting, temperature of the cosmic microwave background radiation, age, and cosmological constant of the universe. When the cosmological constant is zero, we get back Gamow's relation with a much better coefficient. Otherwise, our theory predicts a value of the cosmological constant 2.0×10^{-56} cm^{-2} when the present values of cosmic microwave background temperature of 2.728 K and age of the universe 14 billion years are taken as input.

Keywords Cosmological constant · Cosmology · Self-gravitating systems

1 Introduction

The most important theory for the origin of the universe is the Big Bang Theory [1] according to which the present universe is considered to have started with a huge explosion from a superhot and a superdense stage. Theoretically one may visualize its starting from a mathematical singularity with infinite density. This also comes from the solutions of the type I and type II form of Einstein's field equations [2]. What follows from all these solutions is that the universe has originated from a point where the scale factor R (to be identified as the radius of the universe) is zero at time $t = 0$, and its derivative with time is taken to be infinite at this time. That is, it is thought that the initial explosion had happened with infinite velocity, although, it is impossible for us to picture the initial moment of the creation of the universe. The accelerated expansion of the universe has been conformed by studying the distances to supernovae of type Ia [3, 4]. For the universe, it is being said that the major constituent of the total mass of the present universe is made of the Dark Energy 70%, Dark Matter about 26% and luminous matter 4%. The Dark energy is responsible for the accelerated expansion of

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the universe since it has negative pressure and produces repulsive gravity. The cosmological constant [5, 6] of Einstein provides a repulsive force when its value is positive. The cosmological constant is also associated with the vacuum energy density [7] of the space-time. The vacuum has the lowest energy of any state, but there is no reason in principle for that ground state energy to be zero. There are many different contributions [7] to the ground state energy such as potential energy of scalar fields, vacuum fluctuations as well as of the cosmological constant. The individual contributions can be very large but current observation suggests that the various contributions, large in magnitude but different in sign delicately cancel to yield an extraordinarily small final result. The conventionally defined cosmological constant Λ is proportional to the vacuum energy density ρ_Λ as $\Lambda = (8\pi G/c^2)\rho_\Lambda$. Hence one can guess that $\rho_\Lambda = \Lambda c^2/8\pi G \approx \rho_{Pl} = c^5/G^2\hbar \sim 5\times 10^{93}\ \text{g cm}^{-3}$, where ρ_{Pl} is the Plank density. But the recent observations of the luminosities of high redshift supernovae gives the dimensionless density $\Omega_\Lambda = \rho_\Lambda/\rho_{cr} \equiv \Lambda c^2/3H_0^2 \approx 0.7$ where $\rho_{cr} = 3H_0^2/8\pi G \approx 1.9\times 10^{-29}\ \text{g cm}^{-3}$, which implies $\rho_\Lambda = \rho_{Pl}\times 10^{-123}$. This shows that the cosmological constant today is 123 orders of magnitude smaller. This is known as the 'cosmological constant problem'. In the classical big-bang cosmology there is no dynamical theory [8] to relate the cosmological constant to any other physical variable of the universe. There have been some studies [9–11] regarding the universe to relate the space-time manifold to somekind of condensed matter systems. Here by considering [12, 13] the visible universe made up of self-gravitating particles representing luminous baryons and dark matter such as neutrinos (though only a small fraction) which are fermions and a repulsive potential describing the effect of Dark Energy responsible for the accelerated expansion of the universe, we in this paper derive quantum mechanically a relation connecting temperature, age and cosmological constant of the universe. When the cosmological constant is zero, we get back Gamow's relation with a much better coefficient. Otherwise using as input the current values of $T = 2.728$ K and $t = 14\times 10^9$ years, we predict the value of cosmological constant as $2.0\times 10^{-56}\ \text{cm}^{-2}$. Note that Λ is a completely free parameter in General Theory of Relativity. Also it is interesting to note that we obtain not only the value of the cosmological constant but also the sign of the parameter correct though it is a very small number.

2 Mathematical Formulation without the Cosmological Constant Λ

We in this section derive a relation connecting temperature and age of the universe when cosmological constant is zero, by considering a Hamiltonian [12–14] used by us some time back for the study of a system of self-gravitating particles which is given as:

$$H = -\sum_{i=1}^{N}\left(\frac{\hbar^2}{2m}\right)\nabla_i^2 + \frac{1}{2}\sum_{i=1,}^{N}\sum_{i\neq j, j=1}^{N} v(|\vec{X}_i - \vec{X}_j|) \tag{1}$$

where $v(|\vec{X}_i - \vec{X}_j|) = -g^2/|\vec{X}_i - \vec{X}_j|$, with $g^2 = Gm^2$, G being the universal gravitational constant and m the mass of the effective constituent particles describing the luminous matter and dark matter whose number is $N = \int \rho(\vec{X})d\vec{X}$. Since the measured value for the temperature of the cosmic microwave background radiation is ≈ 2.728 K, it lies in the neighbourhood of almost zero temperature. We, therefore, use the zero temperature formalism for the study of the present problem. Under the situation when N is extremely large, the total kinetic energy of the system is obtained as

$$\langle KE\rangle = \left(\frac{3\hbar^2}{10m}\right)(3\pi^2)^{2/3}\int d\vec{X}[\rho(\vec{X})]^{5/3} \tag{2}$$

where $\rho(\vec{X})$ denotes the single particle density to account for the distribution of particles (fermions) within the system, which is considered to be a finite one. Equation (2) has been written in the Thomas-Fermi approximation. The total potential energy of the system, in the Hartree-approximation, is now given as

$$\langle PE \rangle = -\left(\frac{g^2}{2}\right) \int d\vec{X} d\vec{X}' \frac{1}{|\vec{X} - \vec{X}'|} \rho(\vec{X}) \rho(\vec{X}'). \tag{3}$$

Inorder to evaluate the integral in (2) and (3), we had chosen a trial single-particle density [12, 13] $\rho(\vec{X})$ which is of the form:

$$\rho(\vec{X}) = \frac{Ae^{-x}}{x^3} \tag{4}$$

where $x = (r/\lambda)^{1/2}$, $r = |\vec{X}|$, λ being the variational parameter and A is the normalization constant given as $A = \frac{N}{16\pi\lambda^3}$. Though $\rho(\vec{X})$ is a trial density, still the physics of the behaviour of the density should be incorporated into it while using one. As one can see from (4), $\rho(\vec{X})$ is singular at the origin. This looks to be consistent with the concept behind the Big Bang theory of the universe. The early universe was not only known to be super hot, but also it was superdense. To account for the scenario at the time of the Big Bang, we have, therefore, imagined of a single-particle density $\rho(x)$ for the system which is singular at the origin ($r = 0$). This is only true at the microscopic level, which is not so meaningful looking at things macroscopically. Although $\rho(x)$ is singular, the number of particles N, in the system is finite. Since the present universe has a finite size, its present density which is nothing but an average value is finite. At the time of Big Bang ($t = 0$), since the scale factor (identified as the radius of the universe) is supposed to be zero, the average density of the system can assume an infinitely large value, implying its superdense state. Having thought of a singular form of single-particle density at the time of the Big Bang, we have tried with a number of singular form of single particle densities of the kind $\rho(\vec{r}) = B\frac{\exp[-(\frac{r}{\lambda})^\nu]}{(\frac{r}{\lambda})^{3\nu}}$ where $\nu = 1, 2, 3, 4, \ldots$, or $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$. Though the normalization constant B here is a function of ν, for $\nu = \frac{1}{2}$, $B = A$. Integer values of ν are not permissible because they make the normalization constant infinite. Out of the fractional values, $\nu = \frac{1}{2}$ is found to be most appropriate, because, it has been shown in our earlier paper that it gives the expected upper limit for the critical mass of a neutron star [12] beyond which black hole formation takes place and other parameters of the universe [13] satisfactorily correct. Also because, if ν goes to zero (like $1/n$, $n \to \infty$), $\rho(r)$ would tend to the case of a constant density as found in an infinite many-fermion system. In view of the arguments put forth above, one will have to think that the very choice of our $\rho(r)$ is a kind of ansatz in our theory, which is equivalent to the choice of a trial wave function used in the quantum mechanical calculation for the binding energy of a physical system following variational techniques. As mentioned earlier, singularity at $r = 0$ in the single particle distribution has nothing to do with the average particle density in the system, which happens to be finite (because of the fact that N is finite and volume V of the visible universe is finite), and hence it is not going to affect the large scale spatial homogeneity of the observed universe. Having accepted the value $\nu = \frac{1}{2}$, the parameter λ associated with $\rho(r)$ is determined after minimizing $E(\lambda) = \langle H \rangle$ with respect to λ. This is how, we are able to find the total energy of the system corresponding to its lowest energy state.

After evaluating the integrals in (2) and (3), we find the total energy $E(\lambda)$ of the system which is given as

$$E(\lambda)=\frac{\hbar^2}{m}\frac{12}{25\pi}\left(\frac{3\pi N}{16}\right)^{5/3}\frac{1}{\lambda^2}-\frac{g^2N^2}{16}\frac{1}{\lambda}. \tag{5}$$

We minimize the total energy with respect to λ. Differentiating this with respect to λ and then equating it with zero, we obtain the value of λ at which the minimum occurs. This is found as:

$$\lambda_0=\frac{72}{25}\frac{\hbar^2}{mg^2}\left(\frac{3\pi}{16}\right)^{2/3}\frac{1}{N^{1/3}}. \tag{6}$$

Following the expression for $\langle KE\rangle$ evaluated at $\lambda=\lambda_0$, we write down the value of the equivalent temperature T of the system, using the relation

$$T=\frac{2}{3k_B}\left[\frac{\langle KE\rangle}{N}\right]=\frac{2}{3k_B}(0.015442)N^{4/3}\left(\frac{mg^4}{\hbar^2}\right). \tag{7}$$

The expression for the radius R_0 of the universe, as found by us earlier [12, 13], is given as

$$R_0=2\lambda_0=4.047528\left(\frac{\hbar^2}{mg^2}\right)\Big/N^{1/3}. \tag{8}$$

Our identification of the radius R_0 with $2\lambda_0$ is based on the use of socalled quantum mechanical tunneling [15] effect. Classically, it is well known that a particle has a turning point where the potential energy becomes equal to the total energy. Since the kinetic energy and therefore the velocity are equal to zero at such a point, the classical particle is expected to be turned around or reflected by the potential barrier. From the present theory it is seen that the turning point occurs at a distance $R=2\lambda_0$.

We now invoke Mach's principle [6] which states that inertial properties of matter are determined by the distribution of matter in the rest of the universe. Mach had the view [2] that the velocity and acceleration of a particle would be meaningless had the particle been alone in the universe. We have to talk of acceleration only with respect to other bodies, just like we talk of velocities with respect to other bodies. This means that the inertial mass of a particle is the result of the particle feeling the presence of other particles in the universe. If we denote the inertial mass of the particle by m_{inert}, it is to be determined by its response to accelerated motion. As far as the universe is concerned, the distance particles beyond the Hubble length which we take as the radius of the visible universe R_0 are unobservable and therefore do not contribute to the determination of local inertial mass. If M denotes the gravitational mass of the observable universe, the gravitational energy of the particle is given by $E_{gr}=\frac{GMm_{grav}}{R_0}$, where m_{grav} is the gravitational mass of the particle, that is, the mass determined by its response to gravity. In accordance with the spirit of Mach's principle, one must have $E_{gr}=\frac{GMm_{grav}}{R_0}=m_{inert}c^2$, where $m_{inert}c^2$ is the intrinsic energy of the particle. Since m_{inert} and m_{grav} are taken to be equal both in Newtonian theory and in the General Theory of Relativity, we have Mach's principle [6] expressed through the relation as $(\frac{GM}{R_0c^2})=1$, and using the fact that the total mass of the universe $M=Nm$, we are able to obtain the total number of particles N constituting the universe, as

$$N=2.8535954\left(\frac{\hbar c}{Gm^2}\right)^{3/2}. \tag{9}$$

Table 1 The table describes the age of the universe, temperature as calculated from our theory using (15) and temperature from standard nucleosynthesis calculation

Age of the universe (t) in sec	Temperature (T) in K as calculated from (15)	Temperature (T) in K for the formation of elementary particles [2, 8]
5	$\approx 1\times 10^{9}$	$\approx 6\times 10^{9}$ (e^{+}, e^{-})
1.2×10^{-4}	$\approx 2.1\times 10^{11}$	$\approx 1.2\times 10^{12}$ (μ^{+}, μ^{-} and their antiparticles)
7×10^{-5}	$\approx 2.8\times 10^{11}$	$\approx 1.6\times 10^{12}$ ($\pi^{0}, \pi^{+}, \pi^{-}$ and their antiparticles)
1.5×10^{-6}	$\approx 1.9\times 10^{12}$	$\approx 10^{13}$ (protons, neutron and their antiparticles)
10^{-43}	$\approx 0.73\times 10^{31}$	$\approx 10^{32}$ (Planck mass)

Now, substituting (9) in (8), we arrive at the expression for R_0, as

$$R_0 = 2.8535954\left(\frac{\hbar}{mc}\right)\left(\frac{\hbar c}{Gm^2}\right)^{1/2}. \tag{10}$$

As one can see from above, R_0 is of a form which involves only the fundamental constants like $\hbar$, c, G and the effective mass m which is ofcourse not fundamental. Now, eliminating N from (7), by virtue of (9),we have

$$T = \frac{2}{3}(0.0625019)\left(\frac{mc^2}{k_B}\right). \tag{11}$$

Since we are considering the visible universe, which is actually a patch with a horizon size determined by the speed of light and time that has passed since the Big Bang, we now assume that the radius R_0 of the visible universe is approximately given by the relation

$$R_0 \simeq ct \tag{12}$$

where t denotes the age of the universe at any instant of time. Following (10) and (12), we write m as

$$m = \left(\frac{\hbar^3}{Gc^3}\right)^{1/4}(2.8535954)^{1/2}\frac{1}{\sqrt{t}}. \tag{13}$$

It is interesting to see (as shown in Table 1) this variation of mass with time gives approximately the energy and hence the temperature scale of formation of elementary particles in different epochs of nucleosynthesis. We calculate temperatures in different epochs using our (15) to be derived shortly. This is in good agreement with the calculated values of temperature otherwise known from nucleosynthesis calculations [2, 8]. The period between $t = 7\times 10^{-5}$ s and 5 s is called lepton era, while period before $t = 7\times 10^{-5}$ s is hadron era and the early era corresponding to the period $t < 10^{-43}$ s is known as Planck era. A substitution of m, from (13), in (11), enables us to write

$$T = 0.070388\left(\frac{1}{k_B}\right)\left(\frac{c^5\hbar^3}{G}\right)^{1/4} t^{-1/2} = 0.06339\left[\frac{c^2}{G\, a_B}\right]^{1/4} t^{-1/2}. \tag{14}$$

This is exactly the Gamow's relation [8, 14] apart from the fact that Gamow's relation had the coefficient 0.41563 instead of 0.06339 as in our expression. Substituting the numerical value of a_B, which is equal to 7.56×10^{-15} erg cm^{-3} K^{-4}, and the present value for the universal gravitational constant G [$G = 6.67 \times 10^{-8}$ dyn.cm^2.g^{-2}], in (14), we obtain

$$T = (0.23172 \times 10^{10}) t_{sec}^{-1/2}\ \mathrm{K}. \tag{15}$$

If we accept the age of the universe to be close to 14×10^9 year, which we have used here, with the help of (15), we arrive at a value for the Cosmic Microwave Background Temperature (CMBT) equal to ≈ 3.5 K. This is very close to the measured value of 2.728 K as reported from the most recent Cosmic Background Explorer (COBE) satellite measurements [16–18]. However, if we use Gamow's relation, $t = 956$ billion years is required to obtain the exact value of 2.728 K for the cosmic background temperature. Using our expression, (15), we would require an age of 22.832×10^9 year for the universe to get the exact value of 2.728 K. Long back a correction was made to Gamow's relation by multiplying it with a factor of $(\frac{2}{g_d})^{1/4}$ by taking into account the degeneracies of the particles, where $g_d = 9$. This correction effectively multiplies Gamow'relation with a factor of 0.68 and brings back the age of the universe to 425 billion years for the present CMBT. If we multiply our expression by the same factor to correct for the degeneracy of particles, we obtain a value of 2.4 K, which is less than the value of present CMBT. In the next section we see that by including the cosmic repulsion by the part given by cosmological constant we get back 2.728 K, This is physically correct since the cosmological term [6] has the meaning of negative pressure, it adds energy to the system by its tension when the universe expands, though the over all temperature decreases as the universe expands.

3 Inclusion of the Part of the Hamiltonian Corresponding to the Cosmological Constant

The cosmological constant term [5, 6] Λ associated with vacuum energy density was originally introduced by Einstein as a repulsive component in his field equation and when translated from the relativistic to Newtonian picture gives rise to a repulsive harmonic oscillator force per unit mass as $\sim (\Lambda c^2)\vec{r}$ between points in space when Λ is positive. The one-body operator corresponding to the potential can be written as $H_\Lambda = -\Lambda c^2 |\vec{X}|^2 \rho(\vec{X})$ where $\rho(X)$ here is measured in the unit of mass density. Hence the energy corresponding to this repulsive potential can be written as:

$$\langle H_\Lambda \rangle = -\int \Lambda c^2 |\vec{X}|^2 \rho^2(\vec{X})\, d\vec{X}. \tag{16}$$

By including this contribution of H_Λ in (5), we have the total energy

$$E(\lambda) = \frac{\hbar^2}{m} \frac{12}{25\pi} \left(\frac{3\pi N}{16}\right)^{5/3} \frac{1}{\lambda^2} - \frac{g_\Lambda^2 N^2}{16} \frac{1}{\lambda} \tag{17}$$

where $g_\Lambda^2 = g^2 + \frac{3\Lambda c^2}{16\pi}$. Calculating as before, we have

$$N = 2.8535954 \left(\frac{1}{Gm^2}\right)^{3/4} \left(\frac{\hbar c}{g_\Lambda}\right)^{3/2} \tag{18}$$

and

$$R_0 = 2.8535954 \left(\frac{\hbar}{mc}\right)^{1/2} \left(\frac{\hbar G^{1/4}}{g_\Lambda^{3/2}}\right). \tag{19}$$

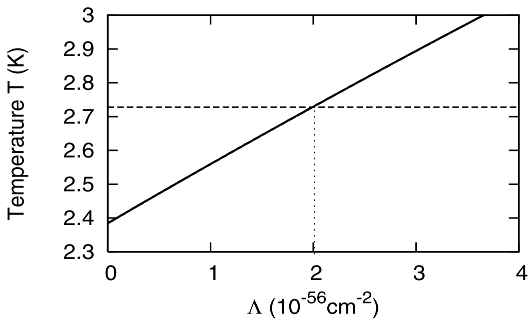

Fig. 1 Determination of Λ by plotting the right hand side of (24) as a function of Λ (*solid line*) and left hand side as 2.728 K (*thin broken line*). The *vertical dotted line* indicates the value of $\Lambda = 2.0\ 10^{-56}\ \text{cm}^{-2}$

Now equating this R_0 with ct we have

$$Gm^{8/3} + \frac{3\Lambda c^2}{16\pi} m^{2/3} - Q = 0 \tag{20}$$

where $Q = \frac{4.0475279\hbar^2 G^{1/3}}{c^2} \times \frac{1}{t^{4/3}}$. Using $m' = m^{2/3}$, the above equation can be cast as a quartic equation in m'. We find [19] four analytic solutions for m' and hence for m. Three of the solutions are unphysical and the only solution which is physically correct is given as

$$m = \left(\frac{u^{1/2} + \sqrt{u - 4(u/2 - [(u/2)^2 + Q/G]^{1/2})}}{2} \right)^{3/2} \tag{21}$$

where

$$u = [r + (q^3 + r^2)^{1/2}]^{1/3} + [r - (q^3 + r^2)^{1/2}]^{1/3} \tag{22}$$

and $r = \frac{9\Lambda^2 c^4}{2(16\pi G)^2}$, $q = \frac{4Q}{3G}$. Now the kinetic energy with the degeneracy factor as discussed in the previous section, is given as

$$T = \left(\frac{2}{g_d}\right)^{\frac{1}{4}} \frac{2}{3k_B} \left[\frac{\langle KE \rangle}{N} \right] = \left(\frac{2}{g_d}\right)^{\frac{1}{4}} \frac{2}{3k_B} (0.015442) N^{4/3} \left(\frac{m g_\Lambda^4}{\hbar^2} \right). \tag{23}$$

Using (18) and (21) in (23), we finally have the relation,

$$T = 0.0417 \left(\frac{2}{g_d}\right)^{1/4} \frac{c^2}{k_B} \frac{[(\{u^{1/2} + \sqrt{4[(u/2)^2 + Q/G]^{1/2} - u}\}/2)^3 + \frac{3\Lambda c^2}{16\pi G}]}{(\{u^{1/2} + \sqrt{4[(u/2)^2 + Q/G]^{1/2} - u}\}/2)^{3/2}}. \tag{24}$$

This is the central result of our paper. This relation connects temperature T with time t and cosmological constant Λ since Q is a function of t and u is also a function of t and Λ. When $\Lambda = 0$, we get back the relation (14) connecting T and t. Since we know the current values of $T = 2.728$ K and $t = 14 \times 10^9$ year, using (24), we solve for Λ. We do that in Fig. 1 by plotting the left hand side and right hand side of (24) and finding the crossing point. This gives $\Lambda = 2.0 \times 10^{-56}\ \text{cm}^{-2}$ which is the value that has been derived dynamically here.

4 Conclusion

To conclude, we in this work have derived a relation connecting temperature, age and cosmological constant of the universe by considering the universe as made up of self-gravitating particles effectively representing luminous matter, dark matter and dark energy represented by the repulsive potential given by the cosmological constant. We have incorporated Mach's Principle into our theory. When the cosmological constant is zero, we get back Gamow's relation with a better coefficient. Other wise our theory predicts the value of cosmological constant as 2.0×10^{-56} cm^{-2}. It is interesting to note that in this flat universe, our method dynamically determines the value of the cosmological constant reasonably well compared to General Theory of Relativity where the cosmological constant is a free parameter.

Acknowledgements We thank B. Mashhoon for his useful comments on the manuscript.

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